OSCILLATIONS

13.1 Which of the following examples represent periodic motion?

(a) A swimmer completing one (return) trip from one bank of a river to the other and back.
(b) A freely suspended bar magnet displaced from its N - S direction and released.
(c) A hydrogen molecule rotating about its centre of mass.
(d) An arrow released from a bow.

SOLUTION:

(a) It is not a periodic motion. Though the motion of a swimmer is to and fro but will not have a definite period.
(b) Since a freely suspended magnet if once displaced from N - S direction and released, it oscillates about this position, it is a periodic motion.
(c) The rotating motion of a hydrogen molecule about its centre of mass is periodic.
(d) Motion of an arrow released from a bow is non-periodic.

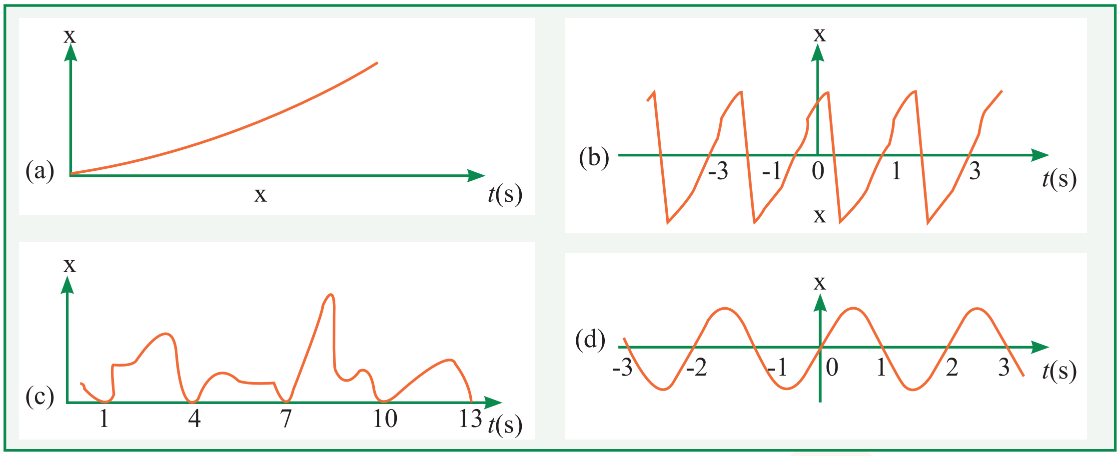
13.2 Which of the following examples represent (nearly) simple harmonic motion and which represent periodic but not simple harmonic motion?

(a) the rotation of earth about its axis.
(b) motion of an oscillating mercury column in a U-tube.
(c) motion of a ball bearing inside a smooth curved bowl, when released from a point slightly above the lower most point.
(d) general vibrations of a polyatomic molecule about its equilibrium position.

SOLUTION:

(a) Since the rotation of earth is not to and fro motion about a fixed point, thus it is periodic but not S.H.M.
(b) It is S.H.M.
(c) It is S.H.M.
(d) General vibrations of a polyatomic molecule about its equilibrium position is periodic but non SHM. In fact, it is a result of superposition of SHMs executed by individual vibrations of atoms of the molecule.

13.3 **Fig. 13.18 depicts four *x* - *t* plots for linear motion of a particle. Which of the plots represent periodic motion? What is the period of motion (in case of periodic motion)?**



SOLUTION:

Figure (b) and (d) represent periodic motions and the time period of each of these is 2 seconds, (a) and (c) are non-periodic motions.

13.4 **Which of the following functions of time represent (a) simple harmonic, (b) periodic but not simple harmonic, and (c) non-periodic motion? Give period for each case of periodic motion (ω is any positive constant):**

(a) **sinωt - cosωt**

(b) **$\sin^3\omega t$**

(c) **3cos (π/4 - 2ωt)**

(d) **cos ωt + cos 3ωt + cos 5ωt**

(e) **$\exp(-\omega^2 t^2)$**

(f) **$1 + \omega t + \omega^2 t^2$**

SOLUTION:

Given –

Need to find –

Answer: The function will represent a periodic motion, if it is identically repeated after a fixed interval of time and will represent S.H.M if it can be written uniquely in the form of a

$\cos\left(\frac{2\pi t}{T}+\phi\right)$ or a $\sin\left(\frac{2\pi t}{T}+\phi\right)$, where T is the time period.

(a) sinωt - cosωt

$$\sin\omega t - \cos\omega t = \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin\omega t - \frac{1}{\sqrt{2}}\cos\omega t\right]$$

$$\sin\omega t - \cos\omega t = \sqrt{2}\left[\sin\omega t\cos\frac{\pi}{4} - \cos\omega t\sin\frac{\pi}{4}\right]$$

$$\sin \omega t - \cos \omega t = \sqrt{2} \sin\left(\omega t - \frac{\pi}{4}\right)$$

It is a S.H.M. and its period is 2π/ω

(b) $\sin^3 \omega t = \frac{1}{3}[3 \sin \omega t - \sin 3\omega t]$

Here each term sin ωt and sin 3ωt individually represent S.H.M. But (ii) which is the outcome of the superposition of two SHMs will only be periodic but not SHMs. Its time period is 2π/ω.

(c) $3\cos\left(\frac{\pi}{4} - 2\omega t\right) = 3\cos\left(2\omega t - \frac{\pi}{4}\right)$ [∵cos (-θ) = cos θ]

Clearly it represents SHM, and its time period is 2π/2ω.

(d) cos ωt + cos3ωt + cos 5ωt. It represents the periodic but not S.H.M. Its time period is 2π/ω

(e) e^{-w2t2}. It is an exponential function which never repeats itself. Therefore it represents non-periodic motion.

(f) $1 + wt + w^2t^2$ also represents non periodic motion.

13.5 A particle is in linear simple harmonic motion between two points, A and B,10 cm apart. Take the direction from A to B as the positive direction and give the signs of velocity, acceleration and force on the particle when it is

(a) at the end A ,

(b) at the end B,

(c) at the mid-point of AB going towards A ,

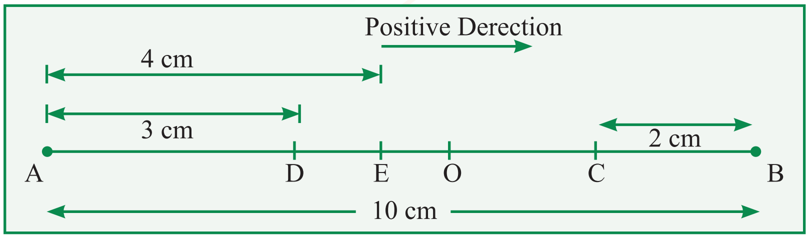
(d) at 2 cm away from B going towards A ,

(e) at 3 cm away from A going towards B, and

(f) at 4 cm away from B going towards A.

SOLUTION:

In the fig. (given below), the points A and B, 10 cm apart, are the extreme positions of the particle in SHM, and the point O is the mean position. The direction from A to B is positive, as indicated.



(a) At the end A, i.e., extreme position, velocity is zero, acceleration and force are directed towards O and are positive.

(b) At the end B, i.e., second extreme position, velocity is zero whereas the acceleration and force are directed towards the point O and are negative.

(c) At the mid-point 0 , while going towards A, velocity is negative and maximum. The acceleration and force both are zero.

(d) At 2 cm away from B, that is, at C and going towards A : v is negative; acceleration and F, being directed towards O , are also negative.

(e) At 3 cm away from A, that is, at D and going towards B : v is positive; acceleration and F, being directed towards O, are also positive.

(f) At a distance of 4 cm away from A and going towards A , velocity is directed along BA, therefore, it is positive. Since acceleration and force are directed towards OB, both are positive.

13.6 Which of the following relationships between the acceleration a and the displacement x of a particle involve simple harmonic motion?

(a) $a = 0.7x$

(b) $a = -200x^2$

(c) $a = -10x$

(d) $a = 100x^3$

SOLUTION:

Only (c) i.e., $a = -10x$ represents SHM. This is because acceleration is proportional and opposite to displacement (x).

13.7 The motion of a particle executing simple harmonic motion is described by the displacement function,

$$x(t) = A\cos(\omega t + \phi)$$

If the initial (t = 0) position of the particle is 1 cm and its initial velocity is ω cm/s, what are its amplitude and initial phase angle? The angular frequency of the particle is πs^{-1}. If instead of the cosine function, we choose the sine function to describe the SHM: $x = B\sin(\omega t + \alpha)$, what are the amplitude and initial phase of the particle with the above initial conditions.

SOLUTION:

Given – $x(t) = A\cos(\omega t + \phi)$, $t = 0$, $x(0) = 1$ cm., $\omega = \pi s^{-1}$

Need to find – amplitude and initial phase angle

Putting x = 1 and t = 0-

$1 = A\cos(\pi \times 0 + \phi$

$A\cos\phi = 1$……………………….(i)

Now differentiating eqn. w.r.t. 't'.

$$v = \frac{d}{dt}x(t) = -A\omega\sin(\omega t + \phi)$$

$\Omega = -A\omega\sin(\pi \times 0 + \phi)$

$A\sin\phi = -1$…………………..(ii)

Squaring and adding eqns (i) and (ii)

$A^2\cos^2\phi + A^2\sin^2\phi = 1^2 + 1^2$.

$A = \sqrt{2}cm$

Dividing eqns. (i) and (ii),

$$\frac{A\sin\phi}{A\cos\phi} = \frac{-1}{1}$$

$$\tan\phi = -1 \Rightarrow \phi = \frac{3\pi}{4}$$

If instead we use the sine function, i.e.,

$$x = B\sin(\omega t + \alpha), \text{ then } v = \frac{d}{dt}B\omega\cos(\omega t + \alpha)$$

At t = 0, using x = 1 and v = ω, we get

1 = Bsin (ω × 0 + α)

Bsin α = …………..(iii)

ω = Bω cos (ω × 0 + α)

Bcos α = 1…………….(iv)

Dividing (v) by (vi),

tan α =1

$$\alpha = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

Squaring (iii) and (iv), we get

$B^2 \sin^2 \alpha + B^2 \cos^2 \alpha = 1^2 + 1$

$$\Rightarrow B = \sqrt{2}cm.$$

13.8 A spring balance has a scale that reads from 0 to 50 kg . The length of the scale is 20 cm. A body suspended from this balance, when displaced and released, oscillates with a period of 0.6 s. What is the weight of the body?

SOLUTION:

Given – M = 50 kg, y = 20 cm = 0.2 m, T = 0.60 ,

$$F = ky \text{ or } Mg = ky \text{ or } k = \frac{Mg}{0.2} = \frac{50\times 9.8}{0.2}Nm^{-1}, K = 2450Nm^{-1}$$

Need to find – weight of the body

Using the formula:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$T^2 = 4\pi^2\frac{m}{k} \text{ \{squaring both sides\}}$$

$$m = \frac{T^2 k}{4\pi^2}$$

$$m = \frac{0.6\times 0.6\times 2460\times 49}{4\times 484}kg = 22.3kg$$

mg = 22.3 × 9.8 N = 218.5 N = 22.3kgf

Hence weight is 22.3kgf.

13.9 A spring having with a spring constant 1200 Nm^{-1} is mounted on a horizontal table as shown in Fig. 13.19. A mass of 3 kg is attached to the free end of the spring. The mass is then pulled sideways to a distance of 2.0 cm and released.

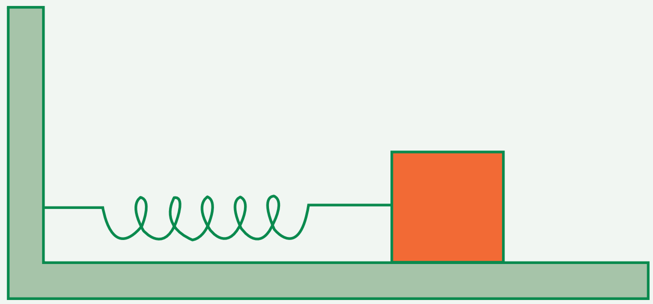


Fig. 13.19

SOLUTION:

Given – spring constant (K)=120 Nm^{-1}; mass(m) = 3.0kg, distance(a) = 2.0 cm = 0.02 m

Need to find – frequency (v), maximum acceleration (A_{max}) , maximum speed (V_{max}).

(i) Frequency,

$$v = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2 \times 3.14}\sqrt{\frac{1200}{3}} = 3.2s^{-1}$$

(ii) Acceleration,

$$A = \omega^2 y = \frac{k}{m} y$$

Acceleration will be maximum when y is maximum i.e., y = a

$$\therefore \text{ max. acceleration, } A_{max} = \frac{ka}{m} = \frac{1200 \times 0.02}{3} = 8ms^{-2}$$

(iii) Max. speed of the mass will be when it is passing through mean position

$$V_{max} = a\omega = a\sqrt{\frac{k}{m}} = 0.02 \times \sqrt{\frac{1200}{3}} = 0.4ms^{-1}$$

13.10 In Exercise 13.9, let us take the position of mass when the spring is unstreched as x = 0, and the direction from left to right as the positive direction of x-axis. Give x as a function of time t for the oscillating mass if at the moment we start the stopwatch (t = 0), the mass is

(a) at the mean position,

(b) at the maximum stretched position, and

(c) at the maximum compressed position.

In what way do these functions for SHM differ from each other, in frequency, in amplitude or the initial phase?

SOLUTION:

Given - $a = 2cm, \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1200}{3}} s^{-1} = 20s^{-1}$

(a) Since time is measured from mean position,
$x = a\sin \omega t = 2\sin 20t$

(b) At the maximum stretched position, the body is at the extreme right position. The initial phase is $\frac{\pi}{2}$.

$$x = a\sin\left(\omega t + \frac{\pi}{2}\right) = a\cos \omega t = 2\cos 20t$$

(c) At the maximum compressed position, the body is at the extreme left position. The initial phase is $\frac{3\pi}{2}$.

$$x = a\sin\left(\omega t + \frac{3\pi}{2}\right) = -a\cos \omega t = -2\cos 20t$$

13.11 Figures 13.20 correspond to two circular motions. The radius of the circle, the period of revolution, the initial position, and the sense of revolution (i.e. clockwise or anti-clockwise) are indicated on each figure.

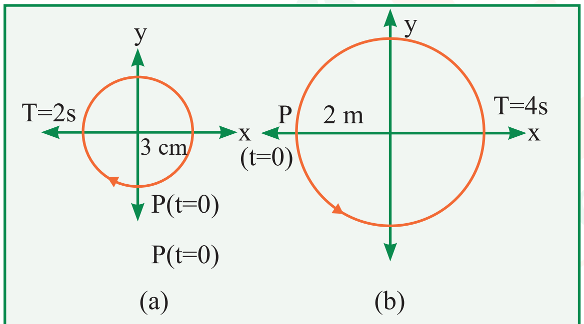


Fig. 13.20

Obtain the corresponding simple harmonic motions of the x-projection of the radius vector of the revolving particle P, in each case.

SOLUTION:

(1) Let A be any point on the circle of reference of the fig. (a) From A, draw BN perpendicular on x-axis. If
$\angle POA = \theta$, then
$\angle OAM = \theta = \omega t$

$\therefore$ In triangle OAM,

$$\frac{OM}{OA} = \sin\theta$$

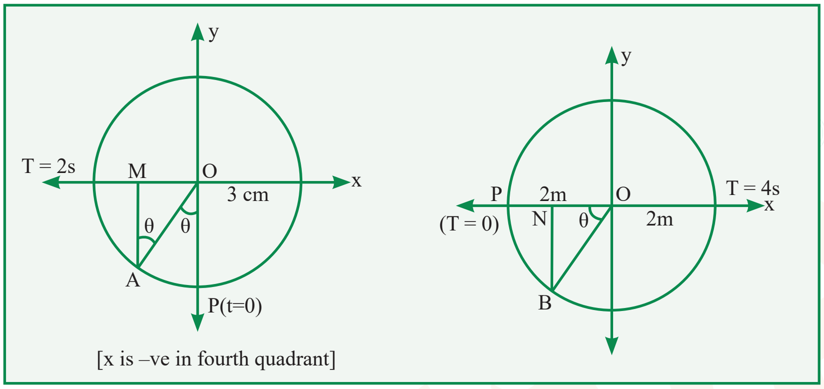
$$\frac{-x}{3} = \sin \omega t = \sin \frac{2\pi}{T} t$$

$$x = -3\sin \frac{2\pi}{2} t$$

$$x = -3\sin \pi t$$

which is the equation of SHM.

(2) Let B be any point on the circle of reference of fig. (b). From B, draw BN perpendicular on x-axis. Then



∠BON = θ = ωt

$\therefore$ In $\Delta ONB, \cos\theta = \frac{ON}{OB}$ ON = OBcosθ

-x = 2cos ωt

$$x = -2\cos\frac{2\pi}{T}t = -2\cos\frac{2\pi}{4}t$$

13.12 Plot the corresponding reference circle for each of the following simple harmonic motions. Indicate the initial (t = 0) position of the particle, the radius of the circle, and the angular speed of the rotating particle. For simplicity, the sense of rotation may be fixed to be anticlockwise in every case: (x is in cm and t is in s).

(a) x = -2sin (3t + π/3)

(b) x = cos (π/6 – t)

(c) x = 3sin (2πt + π/4)

(d) x = 2cos πt

SOLUTION:

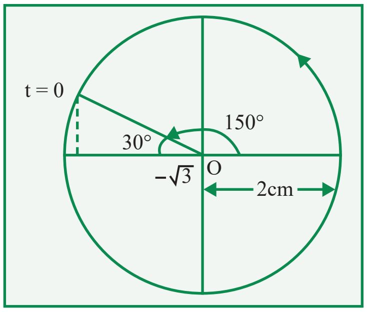
(a) $x = 2\cos\left(3t + \frac{\pi}{3} + \frac{\pi}{2}\right)$

Radius of the reference circle, r = amplitude of SHM =2 cm,

At t = 0, $t = 0, x = -2\sin\frac{\pi}{3} = \frac{-2\sqrt{3}}{2} = -\sqrt{3}cm,$ wt= 3t ∴ ω = 3rad/s

$$\cos\phi_0 = -\frac{\sqrt{3}}{2}, \phi_0 = 150^\circ$$

The reference circle is, thus, as plotted below.



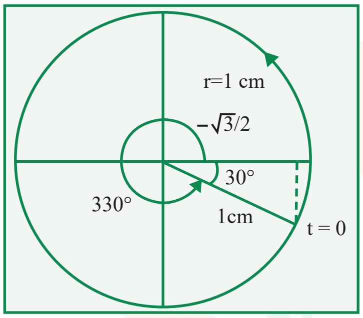
(b) $x = \cos\left(t - \frac{\pi}{6}\right)$

Radius of circle, r = amplitude of SHM = 1 cm.

At $t = 0, x = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} cm, wt = 1t \Rightarrow \omega = 1 rad / s$

$\cos\phi_0 = \frac{\sqrt{3}}{2}, \phi_0 = -\frac{\pi}{6}$

The reference circle is, thus as plotted below



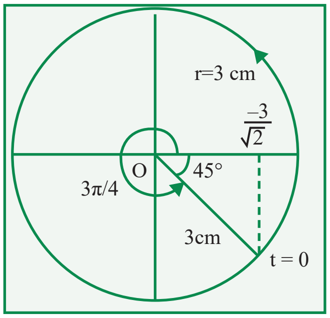
(c) $x = 3\cos\left(2\pi t + \frac{\pi}{4} + \frac{\pi}{2}\right)$

Here is the radius of reference circle, r = 3 cm and at $t = 0, x = 3\sin\frac{\pi}{4} = \frac{\sqrt{3}}{2} cm$

$\omega t = 2\pi t \Rightarrow \omega = 2\pi rad/s$

$\phi_0 = \frac{\sqrt{\frac{3}{2}}}{3} = \frac{1}{\sqrt{2}}$

Therefore, the reference circle is shown below.



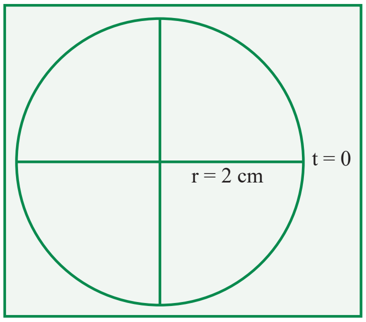
(d) $x = 2\cos \pi t$

Radius of reference circle, r = 2 cm and at t = 0,x = 2 cm

$\omega t = \pi t$, or $\omega = \pi$rad/s

$\cos \phi_0 = 1$, $\phi_0 = 0$

The reference circle is plotted below.



13.13 **Figure 13.21 (a) shows a spring of force constant k clamped rigidly at one end and a mass m attached to its free end. A force F applied at the free end stretches the spring. Figure 13.21 (b) shows the same spring with both ends free and attached to a mass m at either end. Each end of the spring in Fig. 13.21 (b) is stretched by the same force F.**

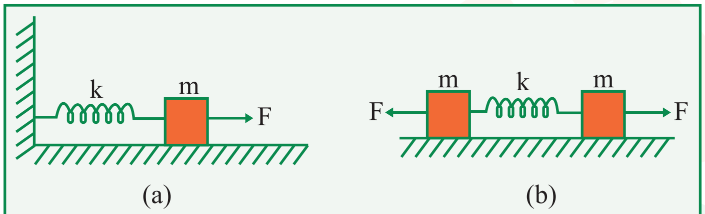


Fig. 13.21

(a) What is the maximum extension of the spring in the two cases ?

(b) If the mass in Fig. (a) and the two masses in Fig. (b) are released, what is the period of oscillation in each case?

SOLUTION:

(a) Let y be the maximum extension produced in the spring in Fig. (a)

Then, F = ky (in magnitude) $\therefore\ y = \frac{F}{k}$

If fig. (b), the force on one mass acts as the force of reaction due to the force on the other mass. Therefore, each mass behaves as if it is fixed with respect to the other.

Therefore,

$$F = ky \Rightarrow y = \frac{F}{k}$$

(b) In fig. (a),

$$\Rightarrow ma = -ky \Rightarrow a = -\frac{k}{m}y \therefore \omega^2 = \frac{k}{m} \text{ i.e., } \omega = \sqrt{\frac{k}{m}}$$

Therefore, period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

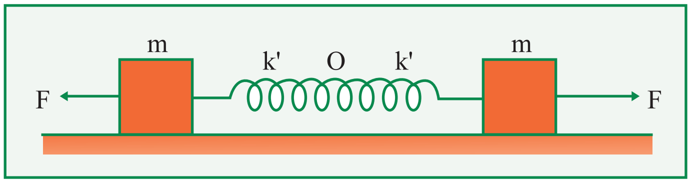
In fig. (b), we may consider that the centre of the system is O and there are two springs each of length $\frac{l}{2}$ attached to the two masses, each m, so that k' is the spring factor of each of the springs.

Then,

$K' = 2k$

$T = 2\pi\sqrt{\frac{m}{k'}}$

$T = 2\pi\sqrt{\frac{m}{2k}}$



13.14 The piston in the cylinder head of a locomotive has a stroke (twice the amplitude) of 1.0 m. If the piston moves with simple harmonic motion with an angular frequency of 200rad/min, what is its maximum speed?

SOLUTION:

Given – Stroke of piston = 2 times the amplitude, A= amplitude, stroke =1 m

Need to find – maximum speed (V_{max})

Answer:

$A = \frac{1}{2}$ m

Angular frequency, ω = 200 rad/min.

We know that the maximum speed of the block when the antplitude is A,

$$V_{max} = \omega A = 200 \times \frac{1}{2} = 100 m/min$$

$$V_{max} = \frac{100}{60} = \frac{5}{3} ms^{-1} = 1.67 ms^{-1}$$

13.15 The acceleration due to gravity on the surface of moon is 1.7 m s^{-2}. What is the time period of a simple pendulum on the surface of moon if its time period on the surface of earth is 3.5 s? (g on the surface of earth is 9.8 m s^{-2})

SOLUTION:

Given –g_m =1.7 ms^{-2}; g_e = 9.8 ms^{-2}; T_e =3.5 s^{-1}

Need to find – time period (T_M)

$$T_e = 2\pi\sqrt{\frac{1}{g_e}} \text{ and } T_m = 2\pi\sqrt{\frac{1}{g_m}}$$

$$\therefore \frac{T_m}{T_e} = \sqrt{\frac{g_e}{g_m}}$$

$$T_m = T_e$$

$$T_m = \sqrt{\frac{g_e}{g_m}}$$

$$T_m = 3.5\sqrt{\frac{9.8}{1.7}} = 8.4\text{s}$$

Hence time period (T_M) is 8.4s

13.16 A simple pendulum of length *I* and having a bob of mass *M* is suspended in a car. The car is moving on a circular track of radius R with a uniform speed v. If the pendulum makes small oscillations in a radial direction about its equilibrium position, what will be its time period?

SOLUTION:

In this case, the bob of the pendulum is under the action of two accelerations.

(i) Acceleration due to gravity 'g' acting vertically downwards.

(ii) Centripetal acceleration $a_c = \frac{v^2}{R}$ acting along the horizontal direction.

$\therefore$ Effective acceleration, $g' = \sqrt{g^2 + a_c^2}$

or

$$g' = \sqrt{g^2 + \frac{v^4}{R^2}}$$

Now time period,

$$T' = 2\pi\sqrt{\frac{1}{g}} = 2\pi\sqrt{\frac{l}{\sqrt{g^2 + \frac{v^4}{R^2}}}}.$$

13.17 A cylindrical piece of cork of density of base area A and height h floats in a liquid of density ρ_r The cork is depressed slightly and then released. Show that the cork oscillates up and down simple harmonically with a period

$$T = 2\pi\sqrt{\frac{h\rho}{\rho_1 g}}$$

where ρ is the density of cork. (Ignore damping due to viscosity of the liquid).

SOLUTION:

Say, initially in equilibrium, y height of cylinder is inside the liquid. Then, Weight of the cylinder = upthrust due to liquid displaced

∴ $Ah\rho g = Ay\rho g$

When the cork cylinder is depressed slightly by Δy and released, a restoring force, equal to additional upthrust, acts on it. The restoring force is

$F = A(y+\Delta y)\rho_l g - Ay\rho_l g = A\rho_l g\,\Delta y$

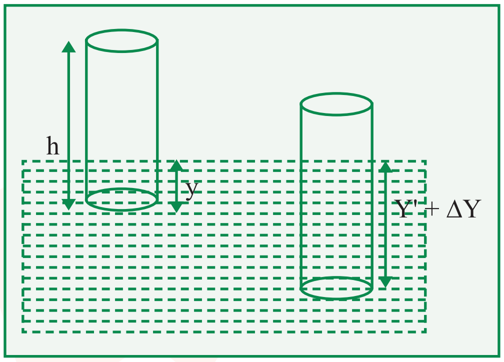
∴ Acceleration, $a = \frac{F}{m} = \frac{A\rho_l g\Delta y}{Ah\rho} = \frac{\rho_l g}{h\rho}.\Delta y$. and the

acceleration is directed in a direction opposite to Δy : Obviously, as a∝ - Δy, the motion of cork cylinder is SHM, whose time period is given by

$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

$$T = 2\pi\sqrt{\frac{\Delta y}{a}}$$

$$T = 2\pi\sqrt{\frac{h\rho}{\rho_l g}}.$$



13.18 One end of a U-tube containing mercury is connected to a suction pump and the other end to atmosphere. A small pressure difference is maintained between the two columns. Show that, when the suction pump is removed, the column of mercury in the U-tube executes simple harmonic motion.

SOLUTION:

The suction pump creates the pressure difference; thus mercury rises in one limb of the U-tube. When it is removed, a net force acts on the liquid column due to the difference in levels of mercury in the two limbs and hence the liquid column executes S.H.M. which can be expressed as:

Consider the mercury contained in a vertical U-tube upto the level P and Q in its two limbs.

Let P = density of the mercury.

L = Total length of the mercury column in both the limbs.

A = internal cross-sectional area of U-tube. m = mass of mercury in U-tube = LAP.

Assume, the mercury be depressed in left limb to F by a small distance y, then it rises by the same amount in the right limb to position Q'.

∴ Difference in levels in the two limbs = P' Q' = 2y.

∴ Volume of mercury contained in the column of length 2y = A×2y

∴ m – A × 2y × ρ.

If W= weight of liquid contained in the column of length 2y .

Then W = mg = A × 2y × ρ × g

This weight produces the restoring force (F) which tends to bring back the mercury to its equilibrium position.

∴ F = – 2Ayρg = – (2Aρg)y

If

a = acceleration produced in the liquid column, Then

$$a = \frac{F}{m}$$

$$a = -\frac{(2A\rho g)y}{LA\rho} = -\frac{2A\rho g}{LA}$$

$$a = -\frac{2\rho g}{2h\rho}y \;\ldots\ldots\ldots\ldots\text{(i)} \qquad (\because L = 2h)$$

where h = height of mercury in each limb. Now from eqn. (i), it is clear that a ∝ y and -ve sign shows that it acts opposite to y, so the motion of mercury in u-tube is simple harmonic in nature having time period (T) given by

$$T = 2\pi\sqrt{\frac{y}{a}} = 2\pi\sqrt{\frac{2h\rho}{2\rho g}} = 2\pi\sqrt{\frac{h\rho}{\rho g}}$$

$$T = 2\pi\sqrt{\frac{h}{g}}$$