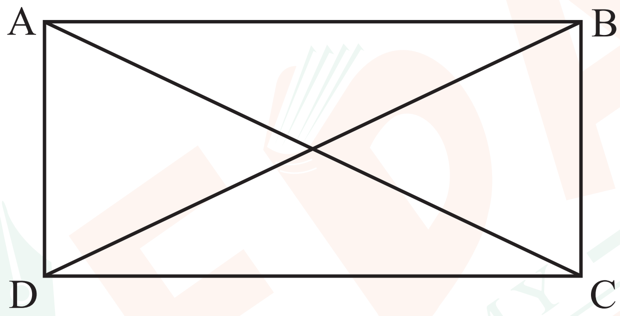
Quadrilaterals

EXERCISE 8.1

1. If the diagonals of a parallelogram are equal, then show that it is a rectangle.

SOLUTION:



To show that ABCD is a rectangle if the diagonals of a parallelogram are equal

To show ABCD is a rectangle, we have to prove that one of its interior angles is right-angled.

Proof, In ΔABC and ΔBAD,

AB = BA (Common)

BC = AD (Opposite sides of a parallelogram are equal)

AC = BD (Given)

Therefore, $\Delta ABC \cong \Delta BAD$ [SSS congruency]

$\angle A = \angle B$ [Corresponding parts of Congruent Triangles]

also,

$\angle A + \angle B = 180°$ (Sum of the angles on the same side of the transversal)

$\Rightarrow 2\angle A = 180°$

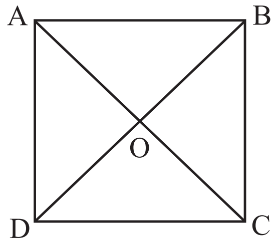
$\Rightarrow \angle A = 90° = \angle B$

Therefore, ABCD is a rectangle.

Hence Proved.

2. Show that the diagonals of a square are equal and bisect each other at right angles.

SOLUTION:



Let ABCD be a square and its diagonals AC and BD intersect each other at O.

To show that, AC = BD

AO = OC

and ∠AOB = 90°

Proof, In ΔABC and ΔBAD,

AB = BA (Common)

∠ABC = ∠BAD = 90° (each 90°)

BC = AD (Opposite side of a square)

ΔABC ≅ ΔBAD [SAS congruency]

Thus,

AC = BD [CPCT]

diagonals are equal.

Now,

In ΔAOB and ΔCOD,

∠BAO = ∠DCO (Alternate interior angles)

∠AOB = ∠COD (Vertically opposite)

AB = CD (opposite side of a square)

ΔAOB ≅ ΔCOD [AAS congruency]

Thus,

AO = CO [CPCT].

Diagonal bisect each other.

Now,

In ΔAOB and ΔCOB,

OB = OB (Common)

AO = CO (diagonals are bisected)

AB = CB (Sides of the square)

ΔAOB ≅ ΔCOB [SSS congruency]

also, ∠AOB = ∠COB [CPCT]

∠AOB+∠COB = 180° (Linear pair)

2∠AOB = 180°

∠AOB = 90°

Diagonals bisect each other at right angles

3. Diagonal AC of a parallelogram ABCD bisects ∠A (see Fig. 8.11). Show that

(i) it bisects ∠C also,

(ii) ABCD is a rhombus.

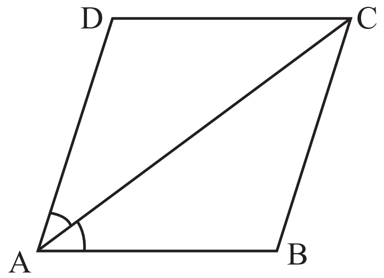


Fig. 8.11

SOLUTION:

(i) ∠DAC = ∠BAC ......(1) [Given]

∠DAC = ∠BCA ......(2) [Alternate angles]

∠BAC = ∠ACD ......(3) [Alternate angles]

From the equations (1), (2) and (3), we have

∠ACD = ∠BCA ......(4)

Hence, diagonal AC bisects angle C also.

(ii) From the equation (2) and (4), we have

∠ACD = ∠DAC

In ΔADC,

∠ACD = ∠DAC [Proved above]

AD = DC [In a triangle, the sides opposite to equal angle are equal]

A parallelogram whose adjacent sides are equal, is a rhombus. Hence, ABCD is a rhombus.

4. ABCD is a rectangle in which diagonal AC bisects ∠A as well as ∠C. Show that:

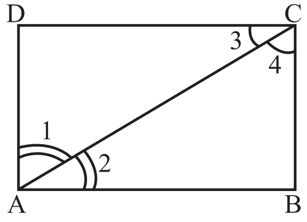
(i) ABCD is a square

(ii) Diagonal BD bisects ∠B as well as ∠D.

SOLUTION:

(i) Given: ABCD is a rectangle ∠1 = ∠2 and ∠3 = ∠4.

To prove: ABCD is a square.



∠1 = ∠4 ......(1) [Alternate angles]

∠3 = ∠4 ......(2) [Given]

Hence, ∠1 = ∠3 ......(3) [From (1) and (2)]

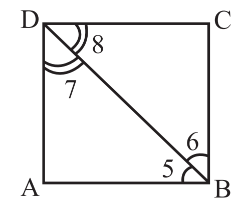
In ΔADC,

∠1 = ∠3 [From (3)]

DC = AD [In a triangle, sides opposite to equal angle are equal]

A rectangle, whose adjacent sides are equal, is a square.

Hence, ABCD is a square.



(ii) To prove: Diagonal BD bisects angle B as well as angle D.

∠5 = ∠8	......(4)	[Alternate angles]
In ΔADB,		
AB = AD		[ABCD is a square]
∠7 = ∠5	......(5)	[Angles opposite to equal sides are equal]
Hence, ∠7 = ∠8	......(6)	[From (4) and (5)]
and ∠7 = ∠6	......(7)	[Alternate angles]
Hence, ∠5 = ∠6	......(8)	[From (5) and (7)]

Hence, from (6) and (8), diagonal BD bisects angle B as well as D.

5. In parallelogram ABCD, two points P and Q are taken on diagonal BD such that DP = BQ (see Fig. 8.12). Show that:

(i) ΔAPD ≅ ΔCQB

(ii) AP = CQ

(iii) ΔAQB ≅ ΔCPD

(iv) AQ = CP

(v) APCQ is a parallelogram

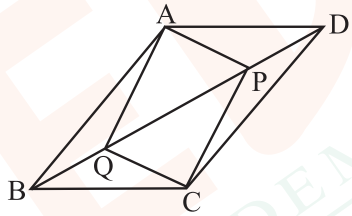


Fig. 8.12

SOLUTION:

(i) In ΔAPD and ΔCQB,
DP = BQ (Given)
∠ADP = ∠CBQ (Alternate interior angles)
AD = BC (Opposite sides of a parallelogram)
Thus, ΔAPD ≅ ΔCQB [SAS congruency]

(ii) AP = CQ by CPCT as ΔAPD ≅ ΔCQB.

(iii) In ΔAQB and ΔCPD,
BQ = DP (Given)
∠ABQ = ∠CDP (Alternate interior angles)
AB = CD (Opposite sides of a parallelogram)
Thus, ΔAQB ≅ ΔCPD [SAS congruency]

(iv) As ΔAQB ≅ ΔCPD
AQ = CP [CPCT]

(v) From the questions (ii) and (iv), the opposite sides of quadrilateral APCQ are equal.
Hence, APCQ is a parallelogram.

6. ABCD is a parallelogram and AP and CQ are perpendiculars from vertices A and C on diagonal BD (see Fig. 8.13). Show that

(i) $\Delta APB \cong \Delta CQD$

(ii) AP = CQ

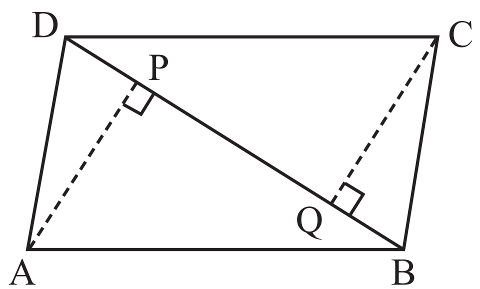


Fig. 8.13

SOLUTION:

(i) In ΔAPB and ΔCQD,

$\angle ABP = \angle CDQ$ (Alternate interior angles)

$\angle APB = \angle CQD$ (equal to 90° as AP and CQ are perpendiculars)

AB = CD (ABCD is a parallelogram)

$\Delta APB \cong \Delta CQD$ [AAS congruency]

(ii) As $\Delta APB \cong \Delta CQD$.

AP = CQ [CPCT]

7. ABCD is a trapezium in which AB || CD and AD = BC (see Fig. 8.14). Show that

(i) $\angle A = \angle B$

(ii) $\angle C = \angle D$

(iii) $\Delta ABC \cong \Delta BAD$

(iv) diagonal AC = diagonal BD

[Hint: Extend AB and draw a line through C parallel to DA intersecting AB produced at E.]

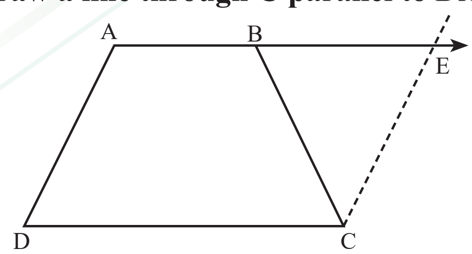


Fig. 8.14

SOLUTION:

(i) **Construction:** Produce AB and draw a line through C parallel to AD, which intersects produced AB at E.

In AECD,

AE || DC [Given]

AD || CE [By construction]

Hence, AECD is a parallelogram.

AD = CE ......(1) [Opposite sides of a parallelogram are equal]

AD = BC ......(2) [Given]

Hence, CE = BC [From the equation (1) and (2)]

Therefore, in ΔBCE,

$\angle 3 = \angle 4$ ......(3) [In a triangle, the angles opposite to equal sides are equal]

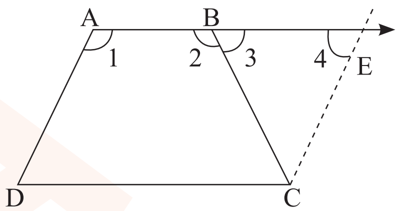
Here, $\angle 2 + \angle 3 = 180°$ ......(4) [Linear Pair]

$\angle 1 + \angle 4 = 180°$ ......(5) [Co-interior angles]

Therefore, $\angle 2 + \angle 3 = \angle 1 + \angle 4$ [From the equation (4) and (5)]

$\Rightarrow \angle 2 = \angle 1 \Rightarrow \angle B = \angle A$ [$\angle 3 = \angle 4$]

(ii) ABCD is a trapezium in which AB || DC, hence,



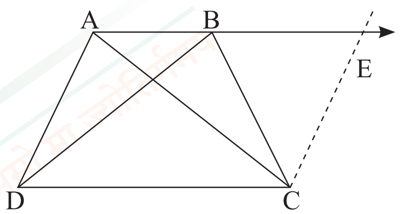
$\angle 1 + \angle D = 180°$ ......(6) [Co-interior angles]

$\angle 2 + \angle C = 180°$ ......(7) [Co-interior angles]

Therefore, $\angle 1 + \angle D = \angle 2 + \angle C$ [From the equation (6) and (7)]

$\Rightarrow \angle D = \angle C$ [$\angle 2 = \angle 1$]

(iii) In ΔABC and ΔBAD,



BC = AD [Given]

$\angle ABC = \angle BAD$ [Proved above]

AB = AB [Common]

Hence, $\Delta ABC \cong \Delta BAD$ [SAS Congruency rule]

(iv) $\Delta ABC \cong \Delta BAD$ [Proved above]

Diagonal AC = Diagonal BD [CPCT]

EXERCISE 8.2

1. ABCD is a quadrilateral in which P, Q, R and S are mid-points of the sides AB, BC, CD and DA (see Fig 8.20). AC is a diagonal. Show that:

(i) **SR || AC and SR = $\frac{1}{2}$ AC**

(ii) **PQ = SR**

(iii) **PQRS is a parallelogram.**

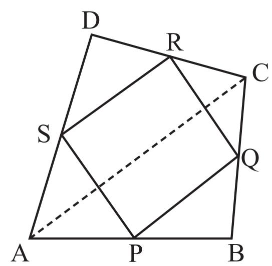


Fig 8.20

SOLUTION:

(i) In ΔDAC,

R is the mid point of DC and S is the mid point of DA.

Thus by mid point theorem, SR || AC and $SR = \frac{1}{2} AC$

(ii) In ΔBAC,

P is the mid point of AB and Q is the mid point of BC.

Thus by mid point theorem, PQ || AC and $PQ = \frac{1}{2} AC$ also, $SR = \frac{1}{2} AC$

PQ = SR

(iii) SR || AC ———————- from question (i)

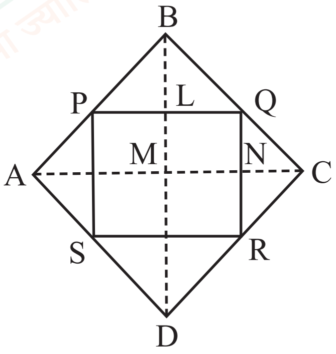
and, PQ || AC ———————- from question (ii)

⇒ SR || PQ – from (i) and (ii) also, PQ = SR

PQRS is a parallelogram.

2. ABCD is a rhombus and P, Q, R and S are the mid-points of the sides AB, BC, CD and DA, respectively. Show that the quadrilateral PQRS is a rectangle.

SOLUTION:



In ΔABC,

P is mid-point of AB [Given]

Q is mid-point of BC [Given]

Hence, PQ || AC and $PQ = \frac{1}{2} AC$ .....(1) [Mid Point Theorem]

Similarly, in ΔACD,

S is mid-point of AD [Given]

R is mid-point of CD [Given]

Hence, SR || AC and $SR = \frac{1}{2} AC$ .....(2) [Mid Point Theorem]

From (1) and (2), we have

PQ || SR .....(3) [PQ || AC and SR || AC]

and PQ = SR .....(4) $\left[SR = \frac{1}{2} AC \text{ and } PQ = \frac{1}{2} AC\right]$

Hence, PQRS is a parallelogram.

Similarly, in ΔBCD,

Q is mid-point of BC [Given]

R is mid-point of CD [Given]

Hence, QR || BD [Mid Point Theorem]

⇒ QN || LM .....(5)

and, LQ || MN .....(6) [PQ || AC]

From (5) and (6), we have

LMNQ is a parallelogram.

Hence, ∠LMN = ∠LQN [Opposite angles of a parallelogram]

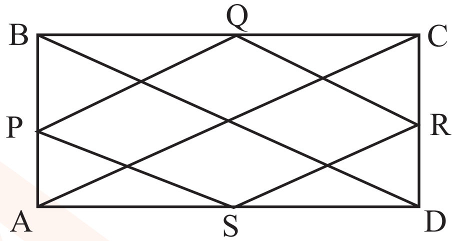
But, ∠LMN = 90° [Digonals of a rhombus are perpendicular to each other]

Hence, ∠LMN = 90°

A parallelogram whose one angle is right angle, is a rectangle. Hence, PQRS is a rectangle.

3. **ABCD is a rectangle and P, Q, R and S are mid-points of the sides AB, BC, CD and DA, respectively. Show that the quadrilateral PQRS is a rhombus.**

SOLUTION:



In ΔABC,

P is mid-point of AB [Given]

Q is mid-point of BC [Given]

Hence, $PQ \parallel AC$ and $PQ = \frac{1}{2}AC$ .....(1) [Mid Point Theorem]

Similarly, in ΔACD,

S is mid-point of AD [Given]

R is mid-point of CD [Given]

Hence, $SR \parallel AC$ and $SR = \frac{1}{2}AC$ .....(2) [Mid Point Theorem]

From (1) and (2), we have

$PQ \parallel SR$ .....(3) [$PQ \parallel AC$ and $SR \parallel AC$]

and PQ = SR .....(4) $\left[SR = \frac{1}{2}AC \text{ and } PQ = \frac{1}{2}AC\right]$

Hence, PQRS is a parallelogram.

Similarly, in ΔBCD,

Q is mid-point of BC [Given]

R is mid-point of CD [Given]

Hence, $QR = \frac{1}{2}BD$ .....(5) [Mid Point Theorem]

Given that: AC = BD .....(6) [Diagonal of a rectangle are equal]

From (1), (5) and (6), we have

PQ = QR

A parallelogram whose adjacent sides are equal, is a rhombus. Hence PQRS is a rhombus.

4. **ABCD is a trapezium in which AB || DC, BD is a diagonal and E is the mid-point of AD. A line is drawn through E parallel to AB intersecting BC at F (see Fig. 8.21). Show that F is the mid-point of BC.**

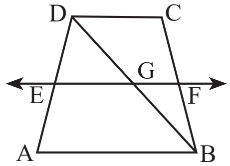


Fig. 8.21

SOLUTION:

In ΔABD,

E is mid-point of AD [Given]

and EG || AB [Given]

Hence, G is mid-point of BD. [Converse of Mid-Point Theorem]

Similarly,

In ΔBCD,

G is mid- point of BD [Proved above]

and FG || DC [Given]

Hence, F is mid-point of BC. [Converse of Mid-Point Theorem]

5. In a parallelogram ABCD, E and F are the mid-points of sides AB and CD, respectively (see Fig. 8.22). Show that the line segments AF and EC trisect the diagonal BD.

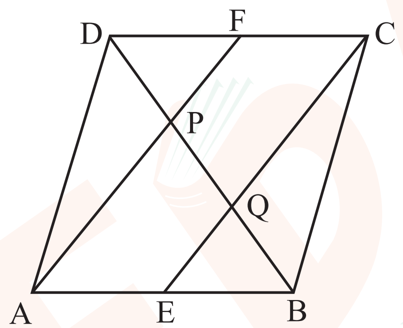


Fig. 8.22

SOLUTION:

In quadrilateral ABCD,

AB = CD [Given]

$\frac{1}{2}AB = \frac{1}{2}CD$

⇒ AE = CF [E and F are the mid-points of AB and CD respectively]

In quadrilateral AECF,

AE = CF [Proved above]

AE || CF [Opposite sides of a parallelogram]

Hence, AECF is a parallelogram.

In ΔDCQ,

F is mid-point of DC [Given]

and FP || CQ [AECF is a parallelogram]

Hence, P is mid-point of DQ [Converse of Mid-Point Theorem]

Hence, DP = PQ .....(1)

Similarly,

In ΔABP,

E is mid-point of AB [Given]
and EQ || AP [AECF is a parallelogram]
Hence, Q is mid-point of PB [Converse of Mid-Point Theorem]
Hence, PQ = QB .....(2)
From (1) and (2), we have
DP = PQ = QB
Hence, line segment AF and EC trisect BD.

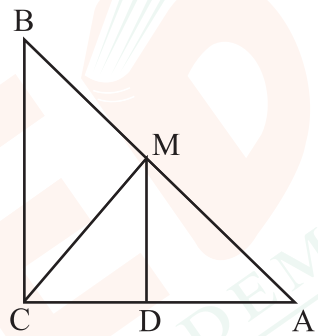
6. ABC is a triangle right angled at C. A line through the mid-point M of hypotenuse AB and parallel to BC intersects AC at D. Show that

(i) D is the mid-point of AC

(ii) MD ⊥ AC

(iii) $CM = MA = \frac{1}{2}AB$

SOLUTION:



(i) In ΔACB,
M is the midpoint of AB and MD || BC
D is the midpoint of AC (Converse of mid point theorem)

(ii) ∠ACB = ∠ADM (Corresponding angles)
also, ∠ACB = 90°
∠ADM = 90° and MD ⊥ AC

(iii) In ΔAMD and ΔCMD,
AD = CD (D is the midpoint of side AC)
∠ADM = ∠CDM (Each 90°)
DM = DM (common)
ΔAMD ≅ ΔCMD [SAS congruency]
AM = CM [CPCT]
also, $AM = \frac{1}{2}AB$ (M is midpoint of AB)
Hence, $CM = MA = \frac{1}{2}AB$